February 1, 2008

ON THE ASYMPTOTIC OF LIKELIHOOD RATIOS FOR SELF-NORMALIZED LARGE DEVIATIONS

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Motivated by multiple statistical hypothesis testing, we obtain the limit of likelihood ratio of large deviations for self-normalized random variables, specifically, the ratio of $P(\sqrt{n}(\bar{X} + d/n) \geq x_n V)$ to $P(\sqrt{n}\bar{X} \geq x_n V)$, as $n \to \infty$, where $\bar{X}$ and V are the sample mean and standard deviation of iid $X_1, \ldots, X_n$, respectively, $d > 0$ is a constant and $x_n \to \infty$. We show that the limit can have a simple form e^{d/z_0}, where z_0 is the unique maximizer of $zf(x)$ with f the density of X_i. The result is applied to derive the minimum sample size per test in order to control the error rate of multiple testing at a target level, when real signals are different from noise signals only by a small shift.

1. Introduction.

1.1. *Background.* Suppose $X_1, X_2, \ldots$ are iid random variables with density f, such that $P(X_1 > 0) > 0$. For $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. We shall consider the biased t statistic

$$T_n = \frac{\sqrt{n}\bar{X}}{V}, \quad \text{with } \bar{X} = \frac{S_n}{n}, \quad V = \left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]^{1/2}.$$

The choice for T_n is only for simplicity of notation. All the results obtained for T_n in the paper hold for the standard t statistic $\sqrt{n-1}\bar{X}/V$ as well.

The aim here is to find the limit of the ratio of tail probabilities for T_n, specifically, the limit of

$$\frac{P\left(\sqrt{n}(\bar{X} + d/n) \geq x_n V\right)}{P\left(\sqrt{n}\bar{X} \geq x_n V\right)}, \qquad \text{as } n \to \infty,$$

where $d > 0$ is a constant and $x_n \to \infty$ in a suitable rate. The problem pertains to large deviations for self-normalized random variables [5, 9]. On

* Research partially supported by NSF grant DMS-0706048 and NIH grant DC007206-01.

AMS 2000 subject classifications: Primary 60F10; secondary 62H15

Keywords and phrases: self-normalize, large deviations, tail probability, multiple hypothesis testing

the other hand, it is directly related to statistical multiple hypothesis testing, in particular, the False Discovery Rate (FDR) control [1], which in recent years has generated intensive research due to its applications in microarray data analysis, medical imagery, etc, where a very large number of signals ("null hypotheses") have to be sorted through in order to identify signals of interest ("false nulls") from the other, noise signals ("true nulls") [6, 7, 8, 10].

A measure of performance for multiple testing is the fraction of falsely identified noise signals ("false discoveries") among the identified ones. Given that at least one signal is identified, the fraction is a well-defined random variable and its conditional expectation is called positive FDR, or pFDR. For a testing procedure, it is desirable that, given a target control level α, the procedure attains pFDR $\le \alpha$. However, whether or not this is possible depends on the property of the data distributions as well as how much data is available to assess the hypotheses. We consider a typical multiple testing problem, where the data distributions are shifted and scaled versions of each other.

Suppose the data distributions are $F_i(x) = F(s_i x - u_i)$, where F is a fixed distribution, and $s_i > 0$ and u_i are unknown. In order to identify from F_i those with $u_i \ne 0$, we test null (hypotheses) $H_i : u_i = 0$ to see which one can be rejected. To this end, let n iid observations be sampled from F_i, which can be written as $Y_{i1} = (X_{i1} + u_i)/s_i, \ldots, Y_{in} = (X_{in} + u_i)/s_i$, with $X_{ij} \sim F$. Suppose the nulls are tested independently of each other, so that X_{ij} are iid for $i \ge 1$, $j = 1, \ldots, n$. Typically, H_i is rejected if and only if the t statistic of $Y_{i1}, \ldots, Y_{in}$ is larger than a cut-off value x_n. Suppose that false nulls occur randomly in the population of nulls, such that each H_i can be false with probability $p \in (0, 1)$ independently of the others, and $u_i = u > 0$ when H_i is false. By definition, a falsely rejected null is a true null, i.e., $u_i = 0$. It is then not hard to see

$$P(H_i \text{ is falsely rejected} \,|\, H_i \text{ is rejected}) = \frac{1-p}{1-p+pR_n}, \tag{1.1}$$

where R_n is the ratio of tail probabilities

$$R_n(u) = \frac{P(\sqrt{n}(\bar{X} + u) \ge x_n V)}{P(\sqrt{n}\bar{X} \ge x_n V)}.$$

It follows that the minimum attainable pFDR is equal to the right hand side of (1.1) as well [4]. Consequently, if real signals are weak in the sense that $u \approx 0$, then R_n can be close to 1, implying that when a nonempty set

of nulls are rejected by whatever multiple testing procedure, it is likely that most or almost all of them are falsely rejected.

For the t test, the only way to address the above limitation on the error rate control is to increase n, the number of observations for each null. From (1.1), in order to attain pFDR $\le \alpha$, n must satisfy

$$R_n(u) \ge (1/p - 1)(1/\alpha - 1). \tag{1.2}$$

An important question is, as $u \approx 0$, what would be the minimum n in order for (1.2) to hold.

The issue of sample size for pFDR control was previously studied in [3]. However, in that work the t statistic was defined in a different way, with $\bar{X}$ and V derived from two independent samples instead of from the same sample. Although that definition allows an easier treatment, it is not commonly used in practice. Furthermore, the asymptotic result in [3] is different from the one reported here for the more commonly used t statistic.

1.2. *Main results.* We need to be more specific about the cut-off value x_n. Usually, as n increases, one can afford to look at more extreme tails to get stronger evidence against nulls. This suggests there should be $x_n \to \infty$ as $n \to \infty$. If $EX > 0$ and $EX^2 < \infty$ for $X \sim F$, then x_n should be at least of the same order as $\sqrt{n}$, otherwise inf pFDR $\to 1$, where the infimum is taken over all possible multiple testing procedures that are solely based on T_i. Furthermore, for $F = N(0,1)$, it is known that there should be $x_n/\sqrt{n} \to \infty$ in order to attain inf pFDR [3]. Based on the considerations, for the general case, we will impose $x_n = a_n\sqrt{n}$ with $a_n \to \infty$ as the cut-off value.

THEOREM 1.1. *Suppose the density f satisfies the following conditions.*

1) f is bounded and continuous on $\mathbb{R}$ and there is $\gamma > 0$, such that

$$\varlimsup_{x\to\infty} x^{1+\gamma} f(x) < \infty.$$

2) $zf(z)$ has a unique maximizer $z_0 > 0$.

3) $h := \log f$ is three times differentiable on $\mathbb{R}$, such that $\sup|h''| < \infty$ and $\sup|h'''| < \infty$.

Let $a_n \to \infty$, such that $a_n^4 = o(n/\log n)$. Then for any $d_n \to d \in (0,\infty)$,

$$\frac{P\left(\bar{X} + d_n/n \ge a_n V\right)}{P\left(\bar{X} \ge a_n V\right)} \to e^{d/z_0}, \quad \text{as} \quad n \to \infty.$$

Note that for different n, $\bar{X}$ and V are different random variables.

Let $k_* = k_*(u)$ be the minimum n in order for (1.2) to hold. The asymptotic of k_* as $u \to 0$ is a consequence of Theorem 1.1.

COROLLARY 1.1. *Suppose f and a_n satisfy the conditions in Theorem 1.1. Let $p \in (0,1)$ and $\alpha \in (0,1)$ be fixed in* (1.2). *Then*

$$k_*(u) \sim (z_0/u)\ln[(1/p-1)(1/\alpha-1)], \qquad \textit{as} \quad u \to 0+.$$

Many probability densities satisfy conditions 1)–3) of Theorem 1.1, for example, Gaussian density $f_1(x;\mu,\sigma) = e^{-(x-\mu)^2/2\sigma^2}/\sqrt{2\pi}\sigma$ and Cauchy density $f_2(x;\mu,\sigma) = \sigma\pi^{-1}[\sigma^2+(x-\mu)^2]^{-1}$. In particular, when $\mu = 0$ and $\sigma = 1$, both have $z_0 = 1$. Therefore, even though all the moments of f_1 are finite whereas all those of f_2 are infinite, in terms of the amount of data needed to control the pFDR, these two are asymptotically the same. On the other hand, Theorem 1.1 is not applicable to densities with zeros on $\mathbb{R}$. Since the conclusion of Theorem 1.1 has nothing to do with the continuity of $h = \log f$ over $\mathbb{R}$, it is desirable to remove condition 3) altogether.

In the rest of the paper, Section 2 proves Theorem 1.1 and Corollary 1.1. Sections 3 and 4 contain proofs of lemmas for the main results.

2. Proof of main results. A key to the proof is the fact that the analysis can be localized at z_0, which is revealed by a representation of the event $\{T_n \ge \sqrt{n}a_n\}$ given by Shao [9]. It is easily seen that for $t > 0$,

$$\{T_n \ge t\} = \left\{\frac{S_n}{Q_n} \ge t\left(\frac{n}{n+t^2}\right)^{1/2}\right\},$$
$$\text{where} \quad Q_n = \sqrt{X_1^2 + \cdots + X_n^2},$$

(cf. [9]). If $t = \sqrt{n}a_n$, then, letting $r = 1 - (1 + a_n^{-2})^{-1/2}$ and following [9],

$$\begin{aligned}
\{T_n \ge \sqrt{n}a_n\} &= \left\{\frac{S_n}{Q_n\sqrt{n}} \ge 1 - r\right\} \\
&= \left\{\sup_{b>0}\sum_{i=1}^n \left[bX_i - \frac{(1-r)}{2}(X_i^2 + b^2)\right] \ge 0\right\} \\
&= \left\{\sup_{b>0}\sum_{i=1}^n \left[\frac{b^2 r(2-r)}{2(1-r)} - \frac{1-r}{2}\left(X_i - \frac{b}{1-r}\right)^2\right] \ge 0\right\} \\
&= \left\{\sup_{b>0}\sum_{i=1}^n \left[\frac{b^2 r(2-r)}{(1-r)^2} - \left(X_i - \frac{b}{1-r}\right)^2\right] \ge 0\right\}.
\end{aligned}$$

Let $z = b/(1-r)$ and $\sigma_n = \sqrt{r(2-r)}$. Then

$$\{T_n \ge \sqrt{n}a_n\} = \left\{\sigma_n^2 \ge \inf_{z>0}\frac{1}{n}\sum_{i=1}^n\left(\frac{X_i}{z} - 1\right)^2\right\}. \tag{2.1}$$

Under the assumption of Theorem 1.1, $r = 1 - (1 + a_n^{-2})^{-1/2} = a_n^{-2}/2 + o(a_n^{-2})$, and hence $\sigma_n^2 \sim 2r = a_n^{-2} + o(a_n^{-2})$, yielding

$$\sigma_n \to 0, \quad n\sigma_n^4/\log n \sim n/(a_n^4 \log n) \to \infty. \tag{2.2}$$

Equations (2.1) and (2.2) are the starting point of the proof.

LEMMA 2.1. *Suppose f satisfies condition 1) and 2) in Theorem 1.1. Let $\sigma_n \to 0$ such that $n\sigma_n^4/\log n \to \infty$. Then, given $r > 0$, there is $\delta = \delta(r) > 0$, such that*

$$\lim_{n\to\infty} \sup_{|d|\le\delta} \frac{P\left\{\sigma_n^2 \ge \inf_{z>0} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\}}{P\left\{\sigma_n^2 \ge \inf_{|z-z_0|\le r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\}} = 1.$$

The lemma will be proved later. The following heuristic explains why the analysis can be localized at z_0. Let $d = 0$. For $\sigma_n^2 \ll 1$, if the event $E_z = \{\sigma_n^2 \ge (1/n)\sum_{i=1}^n (X_i/z - 1)^2\}$ occurs, then most of X_i must fall between $(1 - \sigma_n)z$ and $(1 + \sigma_n)z$, implying

$$\log P(E_z) \approx n \log P(|X - z| \le \sigma_n z) \approx n \log(2\sigma_n z f(z)).$$

As a result, given that at least one E_z occurs, the most likely value of z should be the maximizer of $zf(z)$, i.e., z_0.

The following fact will be used in the proof of Theorem 1.1. If $X_1, \ldots, X_n$ are iid with density f and $n \ge 3$, then the joint density of $\bar{X}$ and V is

$$h(t, s) = (\sqrt{n})^n s^{n-2} \int \prod_{i=1}^n f(t + \sqrt{n} s\omega_i)\mu_n(d\omega) \tag{2.3}$$

where μ_n is the uniform distribution on a $(n - 2)$ dimensional unit sphere perpendicular to $(1, 1, \ldots, 1)$ in $\mathbb{R}^n$, i.e.,

$$U_n := \left\{\omega \in \mathbb{R}^n : \sum_{i=1}^n \omega_i^2 = 1,\ \sum_{i=1}^n \omega_i = 0\right\}.$$

For completeness, a sketch of the proof of (2.3) is given in the Appendix.

Finally, recall that for any $a \in \mathbb{R}$ and random variables $\xi_1, \ldots, \xi_n$,

$$\frac{1}{n}\sum_{i=1}^n (\xi_i - a)^2 = (\bar{\xi} - a)^2 + V_\xi^2,$$

where $\bar{\xi}$ is the sample mean of ξ_i, and $V_\xi = n^{-1/2}\sqrt{\sum_{i=1}^n (\xi_i - \bar{\xi})^2}$ is the biased sample standard deviation.

Proof of Theorem 1.1. Fix $d_n \ge 0$ such that $d_n \to d < \infty$. Given $r > 0$, for $n \gg 1$, $|d_n/n| \le \delta$, where $\delta = \delta(r) > 0$ is as in Lemma 2.1. It therefore suffices to consider the limit of

$$L_n := \frac{P\left\{\sigma_n^2 \ge \inf_{|z-z_0|\le r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_{i,n}}{z} - 1\right)^2\right\}}{P\left\{\sigma_n^2 \ge \inf_{|z-z_0|\le r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\}}$$

where $X_{i,n} := X_i + d_n/n$ has density $f(x - d_n/n)$. Let

$$\Gamma_n = \left\{(t,s) \in (-\infty,\infty)\times[0,\infty) : \sigma_n^2 \ge \inf_{|z-z_0|\le r} \frac{(t-z)^2+s^2}{z^2}\right\}.$$

Then for any random variables $\xi_1,\ldots,\xi_n$,

$$\left\{\sigma_n^2 \ge \inf_{|z-z_0|\le r} \frac{1}{n}\sum_{i=1}^n \left(\frac{\xi_i}{z} - 1\right)^2\right\} = \left\{(\bar\xi, V_\xi) \in \Gamma_n\right\}.$$

Apply the above formula to $X_{i,n}$ and X_i respectively. By (2.1) and (2.3),

$$\begin{aligned} L_n &= \frac{\int_{(t,s,\omega)\in\Gamma_n\times U_n} s^{n-2}\prod_{i=1}^n f(t - d_n/n + \sqrt{n}s\omega_i)\,\mu_n(d\omega)\,dt\,ds}{\int_{(t,s,\omega)\in\Gamma_n\times U_n} s^{n-2}\prod_{i=1}^n f(t + \sqrt{n}s\omega_i)\,\mu_n(d\omega)\,dt\,ds} \\ &= \int_{(t,s,\omega)\in\Gamma_n\times U_n} \rho(t,s,\omega)\nu(dt,ds,d\omega), \end{aligned}$$

where $\nu(dt,ds,d\omega)$ is the probability measure on $\Gamma_n\times U_n$ proportional to $s^{n-2}\prod_{i=1}^n f(t+\sqrt{n}s\omega_i)\,\mu_n(d\omega)\,dt\,ds$, and

$$\rho(t,s,\omega) = \frac{\prod_{i=1}^n f(t - d_n/n + \sqrt{n}s\omega_i)}{\prod_{i=1}^n f(t + \sqrt{n}s\omega_i)}.$$

For each $(t,s,\omega)\in\Gamma_n\times U_n$, by Taylor expansion,

$$\begin{aligned} \rho(t,s,\omega) &= \exp\left\{\sum_{i=1}^n \left[h(t+\sqrt{n}s\omega_i - d_n/n) - h(t+\sqrt{n}s\omega_i)\right]\right\} \\ &= \exp\left\{-\frac{d_n}{n}\sum_{i=1}^n h'(t+\sqrt{n}s\omega_i) + e_n\right\} \end{aligned}$$

where $\sup_{(t,s,\omega)}|e_n| = O(d_n^2/n) = O(1/n)$ due to $\sup_x |h''(x)| < \infty$. By Taylor expansion and $\omega_1 + \cdots + \omega_n = 0$,

$$\frac{1}{n}\sum_{i=1}^n h'(t+\sqrt{n}s\omega_i) = h'(t) + \frac{1}{n}\sum_{i=1}^n h'''(t+\theta\sqrt{n}s\omega_i)(\sqrt{n}s\omega_i)^2$$

for some $\theta \in (0,1)$. Because ω_i^2 add up to 1 and $(t,s) \in \Gamma_n$,

$$\left| \frac{1}{n} \sum_{i=1}^n h'(t + \sqrt{n} s \omega_i) - h'(t) \right| \le \sup_x |h'''(x)| s^2 \le A\sigma_n^2,$$

where $A = (z_0 + r)^2 \sup_x |h'''(x)| < \infty$. For $(t,s) \in \Gamma_n$, as $|t - z| \le \sigma_n z$ for some $z \in [z_0 - r, z_0 + r]$, $|t - z_0| \le r + \sigma_n(z_0 + r) < 2r$ for $n \gg 1$. Combining the above bounds,

$$e^{-d_n \Delta(2r) - A\sigma_n^2 - \sup|e_n|} \le \frac{\rho(t,s,\omega)}{e^{-d_n h'(z_0)}} \le e^{d_n \Delta(2r) + A\sigma_n^2 + \sup|e_n|},$$

with $\Delta(c) = \sup_{|z - z_0| \le c} |h'(z) - h'(z_0)|$. Since r is arbitrary and h' is continuous, from the expression of L_n and $d_n \to d$, it is seen that $L_n \sim e^{-dh'(z_0)}$ as $n \to \infty$. Finally, since z_0 maximizes $\log z + h(z)$, $h'(z_0) = -1/z_0$. So $L_n \sim e^{d/z_0}$. □

Proof of Corollary 1.1. First, it is necessary to show that as $u \to 0+$, $k_*(u) \to \infty$. To this end, it suffices to show that, when n and $c > 0$ are fixed, then

$$\ell(u) := \frac{P(\bar{X} + u \ge cV)}{P(\bar{X} \ge cV)} \to 1, \qquad \text{as} \ \ u \to 0+,$$

where $\bar{X}$ and V are defined in terms of $X_1, \ldots, X_n$. The limit follows from a corollary to Fatou's lemma, which states that if $l_n(x) \le f_n(x) \le u_n(x)$, $l_n(x) \to l(x)$, $f_n(x) \to f(x)$ and $u_n(x) \to u(x)$ pointwise as $n \to \infty$, and $\int l_n \to \int l$ and $\int u_n \to \int u$, then $\int f_n \to \int f$. Specifically, let

$$A(r) = \left\{ (x_1, \ldots, x_n) : r^2 \ge \inf_{z>0} \frac{1}{n} \sum_{i=1}^n \left(\frac{x_i}{z} - 1 \right)^2 \right\}, \qquad \text{for} \ \ r > 0.$$

Then by (2.1), there is $\sigma \in (0,1)$, such that

$$P(\bar{X} + u \ge cV) = \int \mathbf{1}\{x \in A(\sigma)\} \prod_{i=1}^n f(x_i - u)\, dx_1 \cdots dx_n$$
$$P(\bar{X} \ge cV) = \int \mathbf{1}\{x \in A(\sigma)\} \prod_{i=1}^n f(x_i)\, dx_1 \cdots dx_n.$$

Apparently, $0 \le \mathbf{1}\{x \in A(\sigma)\} \prod_{i=1}^n f(x_i - u) \le \prod_{i=1}^n f(x_i - u)$, with the right hand side having the same integral as $\prod_{i=1}^n f(x_i)$. Since $f(x - u) \to f(x)$ pointwise as $u \to 0$, the above corollary to Fatou's lemma implies $P(\bar{X} + u \ge cV) \to P(\bar{X} \ge cV) > 0$. Then $\ell(u) \to 1$.

Next, we show that $uk_*(u)$ is bounded from ∞ as $u \to 0+$. Suppose that there is a sequence u_i such that $u_i k_*(u_i) \to \infty$. Clearly, $n_i := k_*(u_i) \to \infty$. Then, given any M, $u_i n_i \ge M$ for $i \gg 1$ and hence by Theorem 1.1,

$$\begin{aligned}\frac{P(\bar X + u_i \ge a_{n_i} V)}{P(\bar X \ge a_{n_i} V)} &\ge \frac{P(\bar X + M/n_i \ge a_{n_i} V)}{P(\bar X \ge a_{n_i} V)}\\ &\to e^{M/z_0} \gg (1/p-1)(1/\alpha-1),\end{aligned}$$

which contradicts the definition of $k_*(u_i)$.

It only remains to show that $uk_*(u) \to d_0 := z_0 \ln[(1/p-1)(1/\alpha-1)]$ as $u \to 0$. It suffices to show that for any sequence $u_i \to 0$ with convergent $u_i k_*(u_i)$, the limit of $u_i k_*(u_i)$ is d_0. Indeed, let the limit be d. Then, following the above argument, $e^{d/z_0} = (1/p-1)(1/\alpha-1)$, giving $d = d_0$. □

3. Proof of Lemma 2.1.

LEMMA 3.1. Let $\sigma \in (0,1)$, $\eta > 0$ and $s > 0$. Then

$$\begin{aligned}&\left\{\inf_{s\le z\le(1+\eta\sigma)s} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2 \le \sigma^2\right\}\\ &\subset \left\{\frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{s} - 1\right)^2 \le (1+\eta\sigma)^2(1+\eta)^2\sigma^2\right\}.\end{aligned}$$

PROOF. Suppose $(1/n)\sum_{i=1}^n (X_i/z-1)^2 \le \sigma^2$ for some $z \in [s, (1+\eta\sigma)s]$. Then $|\bar X/z - 1| \le \sigma$. By $0 \le 1 - s/z \le \eta\sigma$ and $z^2/s^2 \le (1+\eta\sigma)^2$,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{s}-1\right)^2 &= \frac{1}{ns^2}\sum_{i=1}^n (X_i - \bar X)^2 + \left(\frac{\bar X}{s} - 1\right)^2\\ &= \frac{z^2}{s^2}\left[\frac{1}{nz^2}\sum_{i=1}^n (X_i-\bar X)^2 + \left(\frac{\bar X}{z} - \frac{s}{z}\right)^2\right]\\ &\le \frac{z^2}{s^2}\left[\frac{1}{n}\sum_{i=1}^n\left(\frac{X_i}{z}-1\right)^2 + 2\left|\frac{\bar X}{z} - 1\right|\left(1-\frac{s}{z}\right) + \left(\frac{s}{z}-1\right)^2\right],\end{aligned}$$

with the last expression no greater than $(1+\eta\sigma)^2(1+\eta)^2\sigma^2$. □

In the next, let $X_1, X_2, \ldots$ be iid random variables with density f.

LEMMA 3.2. Suppose $\overline{\lim}_{x\to\infty} x^{1+\gamma} f(x) < \infty$ for some $\gamma > 0$. Let $\sigma_n \to 0$ such that $\underline{\lim}_n n\sigma_n > 0$. Then, given $T > 0$ and $\delta > 0$, there is $a = a(T,\delta) > 0$, such that for $n \gg 1$,

$$\sup_{|d|\le\delta} \frac{1}{n}\log P\left\{\sigma_n^2 \ge \inf_{z\ge a}\frac{1}{n}\sum_{i=1}^n\left(\frac{X_i + d}{z} - 1\right)^2\right\} \le \log\sigma_n - T.$$

PROOF. We first show that there is $a = a(T) > 0$, such that

$$\frac{1}{n} \log P \left\{ \sigma_n^2 \geq \inf_{z \geq a} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i}{z} - 1 \right)^2 \right\} \leq \log \sigma_n - T. \tag{3.1}$$

Fix $\eta \in (0,1)$ with $\eta > (1+\eta/8)^2(1+\eta/4)^2 - 1$. Let $\alpha_n = 1 + \eta\sigma_n/4$. For $n \geq 1$ with $\sigma_n < 1/2$, $\alpha_n < 1 + \eta/8$, so by Lemma 3.1, for any $a > 0$,

$$\begin{aligned} & P \left\{ \sigma_n^2 \geq \inf_{z \geq a} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i}{z} - 1 \right)^2 \right\} \\ & \leq \sum_{j=0}^{\infty} P \left\{ \sigma_n^2 \geq \inf_{a\alpha_n^j \leq z \leq a\alpha_n^{j+1}} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i}{z} - 1 \right)^2 \right\} \\ & \leq \sum_{j=0}^{\infty} P \left\{ (1+\eta)\sigma_n^2 \geq \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i}{a\alpha_n^j} - 1 \right)^2 \right\}. \end{aligned} \tag{3.2}$$

Let $s = (1+\eta)\sigma_n^2$. By Chernoff's inequality, for $z > 0$ and $t > 0$,

$$P \left\{ s \geq \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i}{z} - 1 \right)^2 \right\} \leq \left[z \int e^{ts - tu^2} f(z + zu)\, du \right]^n. \tag{3.3}$$

Fix $A > \overline{\lim}_{x \to \infty} x^{1+\gamma} f(x)$. Let $M(z) = (\gamma/2) \log z$ and $t = M(z)/s$. Then

$$\begin{aligned} z \int e^{ts - tu^2} f(z + zu)\, du & \leq \frac{A}{z^\gamma (1-\eta)^{1+\gamma}} \int_{-\eta}^{\eta} e^{M(z) - M(z)u^2/s}\, du \\ & \quad + z \int_{|u| \geq \eta} e^{M(z) - M(z)\eta^2/s} f(zu + u)\, du \\ & \leq \underbrace{\frac{A e^{M(z)}}{z^\gamma (1-\eta)^{1+\gamma}} \sqrt{\frac{\pi s}{M(z)}}}_{I_1} + \underbrace{e^{M(z) - M(z)\eta^2/s}}_{I_2}. \end{aligned}$$

Since $e^{M(z)} = z^{\gamma/2}$ and $\sqrt{s} = \sqrt{(1+\eta)}\sigma_n$, for $z \gg 1$, $I_1 \leq A z^{-\gamma/2} \sigma_n/2$. On the other hand, $z \gg 1$ and $\sigma_n \ll 1$, the following (in)equalities hold

$$I_2 = z^{\gamma(1-\eta^2/s)/2} \leq z^{-\gamma/2} z^{-\frac{\eta^2}{3(1+\eta)\sigma_n^2}} \leq A z^{-\gamma/2} \sigma_n/2,$$

so $I_1 + I_2 \leq A z^{-\gamma/2} \sigma_n$. Then by (3.2)and (3.3),

$$\begin{aligned} P \left\{ \sigma_n^2 \geq \inf_{z \geq a} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i}{z} - 1 \right)^2 \right\} & \leq \sum_{j=0}^{\infty} \left[A (a\alpha_n^j)^{-\gamma/2} \sigma_n \right]^n \\ & = \frac{(A a^{-\gamma/2} \sigma_n)^n}{1 - (1 + \eta\sigma_n/4)^{-\gamma n/2}} \end{aligned}$$

Since $\sigma_n \to 0$ and $\underline{\lim}_n n\sigma_n > 0$, there is $K > 0$ such that for all $n \gg 1$, $1 - (1 + \eta\sigma_n/4)^{-\gamma n/2} \geq 1 - e^{-\eta\gamma n\sigma_n/9} > 1/K$. Thus

$$\frac{1}{n} \log P\left\{\sigma_n^2 \geq \inf_{z \geq a} \frac{1}{n} \sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\} \leq \log \sigma_n + \log(Aa^{-\gamma/2}) + \frac{\log K}{n}.$$

Since A and K are fixed independently of a, by choosing $a = a(T)$ large enough, (3.1) is proved.

Finally, for $d \in [-\delta, \delta]$ and $z \geq a$,

$$\left(\frac{X_i + d}{z} - 1\right)^2 = \frac{(z-d)^2}{z^2}\left(\frac{X_i}{z-d} - 1\right)^2 \geq \frac{(a-\delta)^2}{a^2}\left(\frac{X_i}{z-d} - 1\right)^2,$$

Therefore,

$$\begin{aligned}
&\sup_{|d| \leq \delta} \frac{1}{n} \log P\left\{\sigma_n^2 \geq \inf_{z \geq a} \frac{1}{n} \sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\} \\
&\leq \frac{1}{n} \log P\left\{\frac{a^2\sigma_n^2}{(a-\delta)^2} \geq \inf_{z \geq a-\delta} \frac{1}{n} \sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\}.
\end{aligned}$$

Then the lemma follows from (3.1). □

LEMMA 3.3. Suppose f is bounded. Let $\sigma_n \to 0$ such that $\underline{\lim}_n n\sigma_n > 0$. Then, given $T > 0$, there is $b = b(T) > 0$, such that for $n \gg 1$,

$$\sup_{d \in \mathbb{R}} \frac{1}{n} \log P\left\{\sigma_n^2 \geq \inf_{0 < z \leq b} \frac{1}{n} \sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\} \leq \log \sigma_n - T.$$

PROOF. Given $\eta > 0$ such that $\eta > (1+\eta/8)^2(1+\eta/4)^2 - 1$, by the same argument for (3.2), for $b > 0$, $d \in \mathbb{R}$ and $n \geq 1$ with $\sigma_n < 1/2$, letting $\alpha_n = 1 + \eta\sigma_n/4$,

$$\begin{aligned}
&P\left\{\sigma_n^2 \geq \inf_{z \leq b} \frac{1}{n} \sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\} \\
&\leq \sum_{j=0}^\infty P\left\{\sigma_n^2 \geq \inf_{b\alpha_n^{-j-1} \leq z \leq b\alpha_n^{-j}} \frac{1}{n} \sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\} \\
&\leq \sum_{j=0}^\infty P\left\{(1+\eta)\sigma_n^2 \geq \frac{1}{n} \sum_{i=1}^n \left(\frac{\alpha_n^j (X_i + d)}{b} - 1\right)^2\right\}.
\end{aligned}$$

Denote $A = \sup f$. For any $s > 0$, $z > 0$ and $d \in \mathbb{R}$,

$$\int e^{-(x/z-1)^2/s} f(x-d)\, dx \leq A \int e^{-(x/z-1)^2/s}\, dx = Az\sqrt{\pi s}.$$

Since the density of $X_i + d$ is $f(x-d)$, by Chernoff's inequality,

$$P\left\{\sigma_n^2 \geq \inf_{z\leq b} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i+d}{z}-1\right)^2\right\} \leq \sum_{j=0}^{\infty}(Ab\alpha_n^{-j}\sqrt{\pi(1+\eta)}\sigma_n)^n$$
$$= \frac{(Ab\sqrt{\pi(1+\eta)}\sigma_n)^n}{1-(1+\eta\sigma_n/4)^{-n}}.$$

By the same argument for Lemma 3.2, the lemma is then proved. □

LEMMA 3.4. Let $0 < b < a < \infty$ and suppose f is continuous and nonzero in a neighborhood of $[b,a]$. If $\sigma_n \to 0$, then, given $\eta > 0$, for $n \gg 1$,

$$\frac{1}{n}\log P\left\{\sigma_n^2 \geq \frac{1}{n}\sum_{i=1}^n\left(\frac{X_i}{z}-1\right)^2\right\} < \log\sigma_n + \log[\sqrt{2\pi e}\, zf(z)] + \eta,$$

holds for all $z \in [b,a]$.

PROOF. Fix $c > 0$ such that $2\log(1+c) < \eta$. Because f is continuous and positive in a neighborhood of $[b,a]$, there is $s > 0$, such that for all $z \in [b,a]$ and $u \in [-s,s]$, $f(z+uz) < (1+c)f(z)$. Then

$$\begin{aligned} I(z) &:= \int \exp\left\{\frac{1}{2} - \frac{1}{2\sigma_n^2}\left(\frac{x}{z}-1\right)^2\right\} f(x)\,dx \\ &= z\sqrt{e}\int e^{-u^2/(2\sigma_n^2)} f(z+uz)\,du \\ &\leq z\sqrt{e}(1+c)f(z)\int_{-s}^{s} e^{-u^2/(2\sigma_n^2)}\,du + ze^{(1-s^2/\sigma_n^2)/2}\int_{|u|\geq s} f(z+uz)\,du \\ &\leq \sqrt{2\pi e}\sigma_n(1+c)zf(z) + e^{(1-s^2/\sigma_n^2)/2}. \end{aligned}$$

By $\inf_{z\in[b,a]} zf(z) > 0$ and $\sigma_n \to 0$, it follows that for $n \gg 1$, $I(z) < \sqrt{2\pi e}\sigma_n(1+c)^2 zf(z) < \sqrt{2\pi e}\sigma_n e^{\eta} zf(z)$. Together with Chernoff's inequality, this implies the inequality in the lemma. □

To demonstrate Lemma 2.1, we need the following application of the uniform exact LDP of [2]. The result will be proved in the next section.

PROPOSITION 3.1. Suppose f is bounded on $\mathbb{R}$. Let $z > 0$ such that f is continuous and nonzero at z. Define

$$h(t) = \log\left[z\int e^{-tu^2} f(z+uz)\,du\right], \quad t \geq 0. \tag{3.4}$$

Let $\sigma_n \to 0$ such that $n\sigma_n^4/\log n \to \infty$. Then, for each n, there is a unique $t_n > 0$, such that $h'(t_n) = -\sigma_n^2$, and moreover, as $n \to \infty$,

$$t_n \sim \frac{1}{2\sigma_n^2}, \tag{3.5}$$

$$h(t_n) = \log \sigma_n + \log[\sqrt{2\pi}\, z f(z)] + o(1) \tag{3.6}$$

$$P\left\{\sigma_n^2 \ge \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\} \sim \frac{\exp\{n(\sigma_n^2 t_n + h(t_n))\}}{\sqrt{\pi n}}. \tag{3.7}$$

PROOF OF LEMMA 2.1. It suffices to show that there is $\delta = \delta(r) > 0$, such that

$$\lim_{n\to\infty} \sup_{|d|\le\delta} \frac{P\left\{\sigma_n^2 \ge \inf_{z>0,\,|z-z_0|>r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i+d}{z} - 1\right)^2\right\}}{P\left\{\sigma_n^2 \ge \inf_{|z-z_0|\le r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i+d}{z} - 1\right)^2\right\}} = 0.$$

Denote the denominator by $B(r,d)$. Given $0 < \eta \ll 1$, when $|d| \ll \min(r, z_0)$,

$$\begin{aligned}
B(r,d) &= P\left\{\sigma_n^2 \ge \inf_{|z-z_0|\le r} \frac{1}{n}\left(\frac{z-d}{z}\right)^2 \sum_{i=1}^n \left(\frac{X_i}{z-d} - 1\right)^2\right\} \\
&\ge P\left\{(1-\eta)\sigma_n^2 \ge \inf_{|z-z_0|\le r-d} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\} \\
&\ge P\left\{(1-\eta)\sigma_n^2 \ge \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z_0} - 1\right)^2\right\}.
\end{aligned}$$

Therefore, it is enough to show that there is $\delta > 0$, such that

$$\lim_{n\to\infty} \frac{\sup_{|d|\le\delta} P\left\{\sigma_n^2 \ge \inf_{z>0,\,|z-z_0|>r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i+d}{z} - 1\right)^2\right\}}{P\left\{(1-\eta)\sigma_n^2 \ge \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z_0} - 1\right)^2\right\}} = 0. \tag{3.8}$$

By the assumption of the lemma,

$$D := \log[z_0 f(z_0)] - \sup_{z>0,\,|z-z_0|\ge r/2} \log[z f(z)] > 0.$$

By Proposition 3.1, as long as $\eta > 0$ is small enough, as $n \to 0$,

$$\begin{aligned}
&\frac{1}{n}\log P\left\{(1-\eta)\sigma_n^2 \ge \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z_0} - 1\right)^2\right\} \\
&= (1-\eta)\sigma_n^2 t_n + \log(\sqrt{1-\eta}\,\sigma_n) + \log[\sqrt{2\pi}\, z f(z)] + o(1) \\
&\ge M_n := \log\sigma_n + \log[\sqrt{2\pi e} z_0 f(z_0)] - D/4. \qquad (3.9)
\end{aligned}$$

Since $\sigma_n \to 0$ and $n\sigma_n^4/\log n \to \infty$, $n\sigma_n \to \infty$ as well. By Lemmas 3.2 – 3.3, there are $b \in (0, z_0 - r)$, $a \in (z_0 + r, \infty)$ and $\delta_0 > 0$, such that

$$\sup_{|d|\le\delta_0} \frac{1}{n} \log P\left\{\sigma_n^2 \ge \inf_{z\notin[b,a]} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\} \le M_n - D/2, \tag{3.10}$$

Fix $0 < \delta \le \delta_0$ such that $\delta < \min(r/2, b/2, a)$ and $z^2 < (1+\eta)(z-\delta)^2$ for all $z \in [b, a]$. Then

$$\begin{aligned}
&\sup_{|d|\le\delta} P\left\{\sigma_n^2 \ge \inf_{b\le z\le a,\, |z-z_0|>r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\}\\
&= \sup_{|d|\le\delta} P\left\{\sigma_n^2 \ge \inf_{b\le z\le a,\, |z-z_0|>r} \frac{1}{n}\left(\frac{z-d}{z}\right)^2 \sum_{i=1}^n \left(\frac{X_i}{z-d} - 1\right)^2\right\}\\
&\le P\left\{\sup_{b\le z\le a,\, |d|\le\delta} \left(\frac{z}{z-d}\right)^2 \sigma_n^2 \ge \inf_{b-\delta\le z\le a+\delta,\, |z-z_0|\ge r/2} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\}\\
&\le P\left\{(1+\eta)\sigma_n^2 \ge \inf_{z\in J} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\},
\end{aligned}$$

where $J = [b/2, z_0 - r/2] \cup [z_0 + r/2, 2a]$. By Lemma 3.4 and the definition of D, as long as η is chosen small enough, for all $n \gg 1$,

$$\sup_{z\in J} \frac{1}{n} \log P\left\{(1+\eta)^4\sigma_n^2 \ge \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\} \le M_n - D/2.$$

Let $\alpha_n = 1 + \eta\sigma_n$ and $N(n) = \lceil \log(4a/b)/\log\alpha_n \rceil$. It is not hard to see that J can be covered by the union of at most $N(n)$ intervals of the form $I_k = [x_k, \alpha_n x_k]$. By Lemma 3.1 and the above inequality, for $n \gg 1$,

$$\begin{aligned}
&P\left\{(1+\eta)\sigma_n^2 \ge \inf_{z\in J} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\}\\
&\le \sum_k P\left\{(1+\eta)\sigma_n^2 \ge \inf_{z\in I_k} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\}\\
&\le N(n) \max_k P\left\{(1+\eta)^4\sigma_n^2 \ge \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{\alpha_n x_k} - 1\right)^2\right\}
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{|d|\le\delta} \frac{1}{n} \log P\left\{\sigma_n^2 \ge \inf_{b\le z\le a,\, |z-z_0|>r} \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i + d}{z} - 1\right)^2\right\}\\
&\le M_n - \frac{D}{2} + \frac{\log N(n)}{n}.
\end{aligned}\tag{3.11}$$

As $n \to \infty$, $N(n) \sim \log(4a/b)/(\eta\sigma_n) = O(\sigma_n^{-1})$. Since $n\sigma_n^4/\log n \to \infty$, $\log N(n)/n \to 0$. By combining (3.9) – (3.11), (3.8) is thus proved. ☐

4. Proof of Proposition 3.1. Given $z > 0$, the log-moment generating function of $-(X/z-1)^2$ is $h(t)$, which is defined in (3.4). It is not hard to see that for $t \ge 0$, $h(t) < \infty$ and

$$h'(t) = -\frac{\int u^2 e^{-tu^2} f(z+uz)\,du}{\int e^{-tu^2} f(z+uz)\,du} < 0, \tag{4.1}$$

$$h''(t) = \frac{\int u^4 e^{-tu^2} f(z+uz)\,du}{\int e^{-tu^2} f(z+uz)\,du} - [h'(t)]^2 > 0. \tag{4.2}$$

Lemma 4.1. Fix $z > 0$. Suppose f is continuous and nonzero at z. Also suppose $\sup f < \infty$. Then for $p > -1$,

$$\int |u|^p e^{-tu^2} f(z+uz)\,du \sim f(z)\Gamma(p')t^{-p'}, \quad \text{as } \ t \to \infty,$$

with $p' = (p+1)/2$.

Proof. Given $c > 1$, there is $\eta > 0$, such that $f(z)/c < f(z+uz) < cf(z)$ for all $u \in [-\eta, \eta]$. Write

$$\int |u|^p e^{-tu^2} f(z+uz)\,du = \int_{-\eta}^{\eta} + \int_{|u|>\eta} = I_1 + I_2.$$

By the selection of c,

$$\frac{f(z)}{c}\int_{-\eta}^{\eta} |u|^p e^{-tu^2}\,du < I_1 < cf(z)\int_{-\eta}^{\eta} |u|^p e^{-tu^2}\,du.$$

As $t \to \infty$,

$$\int_{-\eta}^{\eta} |u|^p e^{-tu^2}\,du \sim \int |u|^p e^{-tu^2}\,du = \Gamma(p')t^{-p'},$$
$$I_2 \le \sup f \int_{|u|>\eta} |u|^p e^{-tu^2}\,du = o(t^{-p'}).$$

Since $c > 1$ is arbitrary, the lemma is proved. □

Proof of Proposition 3.1. Because $h'' > 0$ on $(0,\infty)$, h' is strictly increasing on $(0,\infty)$. By Lemma 4.1, $h'(t) \sim -(2t)^{-1}$ as $t \to \infty$. Thus, by $\sigma_n \to 0$, for $n \gg 1$, there is a unique $t_n \to \infty$ with $\sigma_n^2 = -h'(t_n) \sim (2t_n)^{-1}$. This proves (3.5). By Lemma 4.1,

$$h(t_n) = \log\left[z\int e^{-t_n u^2} f(z+uz)\,du\right] = \log[\,(1+o(1))zf(z)\sqrt{\pi/t_n}\,\,].$$

Together with (3.5), this implies (3.6).

It remains to show (3.7). For large n, t_n is well-defined. Because $\sigma_n^2 t + h(t)$ is strictly convex, $t_n = \arg\inf_{t>0}[\sigma_n^2 t + h(t)]$. Let

$$f_n(x) = e^{-t_n(x/z-1)^2 - h(t_n)} f(x).$$

It is seen that f_n is a probability density. Let $\xi_{nk} = -(\zeta_{nk}/z - 1)^2$, where ζ_{nk} are iid with density f_n. Then by (4.1)

$$E(\xi_{nk}) = -\int (x/z-1)^2 f_n(x) = -z\int u^2 e^{-t_n u^2 - h(t_n)} f(z+zu)\, du = h'(t_n)$$

and likewise by (4.2), $\mathrm{Var}(\xi_{nk}) = h''(t_n)$. Define

$$\begin{cases} Y_n = \dfrac{\xi_{n1} + \ldots + \xi_{nn} - nh'(t_n)}{\sqrt{nh''(t_n)}}, \\ T_n = -\displaystyle\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2, \quad G_n(t) = E[e^{tT_n}], \quad t > 0 \end{cases} \tag{4.3}$$

and $\Lambda_n(t) = \log G_n(t)$. By checking the characteristic function of $\xi_{n1} + \ldots + \xi_{nn}$, it can be seen that Y_n also has the representation

$$Y_n \sim \frac{\tilde{T}_n - \Lambda_n'(t_n)}{\sqrt{\Lambda_n''(t_n)}}, \quad \text{with} \quad P(\tilde{T}_n \in dx) = e^{t_n x - \Lambda_n(t_n)} P(T_n \in dx), \tag{4.4}$$

and hence characteristic function

$$E[e^{itY_n}] = \exp\left\{-\frac{it\Lambda_n'(t_n)}{\sqrt{\Lambda_n''(t_n)}}\right\} G_n\left(t_n + \frac{it}{\sqrt{\Lambda_n''(t_n)}}\right) \Big/ G_n(t_n). \tag{4.5}$$

Since $\Lambda_n(t) = nh(t)$, then $\Lambda_n'(t_n) = -n\sigma_n^2$ and, by Lemma 4.1 and (3.5),

$$\Lambda_n''(t_n) = nh_n''(t_n) \sim n/(2t_n^2) \sim 2n\sigma_n^4, \quad \text{as} \quad n \to \infty. \tag{4.6}$$

By standard exponential tilting,

$$\begin{aligned} P\left\{\sigma_n^2 \geq \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{z} - 1\right)^2\right\} &= P\left\{T_n \geq -n\sigma_n^2\right\} \\ &= e^{n\sigma_n^2 t_n + \Lambda_n(t_n)} E\left[\mathbf{1}\{Y_n \geq 0\}\, e^{-t_n\sqrt{\Lambda_n''(t_n)}Y_n}\right] \end{aligned}$$

Therefore, in order to show (3.7), it suffices to show

$$E\left\{\mathbf{1}\{Y_n \geq 0\}\, e^{-t_n\sqrt{\Lambda_n''(t_n)}Y_n}\right\} \sim \frac{1}{t_n\sqrt{2\pi\Lambda_n''(t_n)}}. \tag{4.7}$$

The proof is based on the next lemma, which is essentially established in [2].

LEMMA 4.2. For each n, let T_n be a random variable such that $G_n(t) = E[e^{tT_n}] < \infty$ in a neighborhood of $t_n \in \mathbb{R}$. Let $\Lambda_n(t) = \log G_n(t)$ and Y_n be defined as in (4.4). Suppose that, as $n \to \infty$,

$$\Lambda_n''(t_n) \to \infty, \quad t_n^2 \Lambda_n''(t_n) \to \infty, \tag{4.8}$$

$$Y_n \xrightarrow{d} N(0,1), \tag{4.9}$$

and there is $\delta > 0$ and $n_0 \geq 1$, such that

$$f^*(t) := \sup_{n \geq n_0} \left| E[e^{itY_n}] \mathbf{1}\left\{ |t| \leq \delta \sqrt{\Lambda_n''(t_n)} \right\} \right| \in L^1, \tag{4.10}$$

$$\sup_{\delta < |y| \leq \lambda t_n} \left| \frac{G_n(t_n + iy)}{G_n(t_n)} \right| = o\left(\frac{1}{t_n \sqrt{\Lambda_n''(t_n)}} \right), \forall \lambda > 0. \tag{4.11}$$

Then (4.7) holds.

PROOF. Let $\beta_n = \delta \sqrt{\Lambda_n''(t_n)}$ and $b_n = t_n \sqrt{\Lambda_n''(t_n)}$. Then by (4.5), the characteristic function of Y_n satisfies conditions (2.7) and (2.8) of Theorem 2.3 in [2], and hence (2.9) and (2.10) there. Then by $Y_n \to N(0,1)$ and Theorem 2.7 in [2], (4.7) follows. $\square$

Continuing the proof of Proposition 3.1, it suffices to verify (4.8) – (4.11) for T_n defined in (4.3). By (4.6) and the assumption that $n\sigma_n^4/\log n \to \infty$, (4.8) is clear. To show (4.9), consider the representation in (4.3). Because $EY_n = 0$ and $\mathrm{Var}(Y_n) = 1$, we only need to check the Lindeberg condition, i.e., for any $a > 0$,

$$nE\left[\left(\frac{\xi_n - h'(t_n)}{\sqrt{nh''(t_n)}} \right)^2 \mathbf{1}\left\{ \left| \frac{\xi_n - h'(t_n)}{\sqrt{nh''(t_n)}} \right| \geq a \right\} \right] \to 0, \quad \text{with } \xi_n \sim \xi_{nk}.$$

Since $h'(t_n) = \sigma_n^2$ and $h''(t_n) \sim 2\sigma_n^4$, for $n \gg 1$, $|\xi_n - \sigma_n^2| \geq a\sqrt{nh''(t_n)}$ implies $|\xi_n| \geq a\sqrt{n}\sigma_n^2$ and $|\xi_n - \sigma_n^2| \leq 2|\xi_n|$. It thus suffices to show

$$E\left[\xi_n^2 \mathbf{1}\left\{ |\xi_n| \geq a\sqrt{n}\sigma_n^2 \right\} \right] = o(\sigma_n^4). \tag{4.12}$$

By the definition of f_n, the expectation on the left hand side is equal to

$$\begin{aligned} &e^{-h(t_n)} \int_{(x/z-1)^2 \geq a\sqrt{n}\sigma_n^2} \left(\frac{x}{z} - 1 \right)^4 e^{-t_n(x/z-1)^2} f(x)\, dx \\ &= ze^{-h(t_n)} \int_{u^2 \geq a\sqrt{n}\sigma_n^2} u^4 e^{-t_n u^2} f(z + zu)\, du \leq (z \sup f) e^{-h(t_n)} I_n, \end{aligned}$$

where, by change of variable $u = x/\sqrt{t_n}$ and $\sigma_n^2 t_n \sim 1/2$,

$$I_n := \int_{u^2 \ge a\sqrt{n}\sigma_n^2} u^4 e^{-t_n u^2}\, du \le t_n^{-5/2} \int_{x^2 \ge b\sqrt{n}} x^4 e^{-x^2}\, dx, \quad \text{for } n \gg 1,$$

with $b \in (0, a/2)$ a constant. Since

$$\int_{x^2 \ge b\sqrt{n}} x^4 e^{-x^2}\, dx = \int_{b\sqrt{n}}^{\infty} y^{3/2} e^{-y}\, dy \sim (b\sqrt{n})^{3/2} e^{-b\sqrt{n}} = o(1),$$

$I_n = o(t_n^{-5/2}) = o(\sigma_n^5)$. On the other hand, by Lemma 4.1, $e^{-h(t_n)} \sim 1/(zf(z)\sqrt{2\pi}\sigma_n) = O(\sigma_n^{-1})$. As a result, $e^{-h(t_n)} I_n = o(\sigma_n^4)$, yielding (4.12).

To show (4.10) and (4.11), notice that

$$\left|\frac{G_n(t_n + iy)}{G_n(t_n)}\right| = |\phi_n(y)|^n$$
$$\text{where} \quad \phi_n(y) = z e^{-h(t_n)} \int e^{-t_n u^2 - iyu^2} f(z + zu)\, du.$$

Fix $0 < c \ll 1$. Since f is continuous and nonzero at z, there is $r \in (0, 1/2)$ such that $f(z)/(1+c) \le f(z+uz)$ and $1 - u^2 \le \cos u \le 1 - u^2/(2+c)$ for $u \in [-r, r]$. Write

$$\phi_n(y) = z e^{-h(t_n)} \int_{|u| \le \sqrt{r/y}} + z e^{-h(t_n)} \int_{|u| > \sqrt{r/y}} = I_n(y) + J_n(y).$$

Then for $n \gg 1$ and $y \in \mathbb{R}$,

$$\begin{aligned}
|\mathrm{Re}\, I_n(y)| &= z e^{-h(t_n)} \int_{-\sqrt{r/y}}^{\sqrt{r/y}} \cos(yu^2) e^{-t_n u^2} f(z + zu)\, du \\
&\le 1 - \frac{z e^{-h(t_n)}}{2 + c} \int_{-\sqrt{r/y}}^{\sqrt{r/y}} y^2 u^4 e^{-t_n u^2} f(z + zu)\, du \\
&\le 1 - \frac{z e^{-h(t_n)} y^2}{2 + c} \int_{-\sqrt{r/(y\vee 1)}}^{\sqrt{r/(y\vee 1)}} u^4 e^{-t_n u^2} f(z + zu)\, du \\
&\le 1 - \frac{z e^{-h(t_n)} y^2 f(z)}{(1 + c)(2 + c)} \int_{-\sqrt{r/(y\vee 1)}}^{\sqrt{r/(y\vee 1)}} u^4 e^{-t_n u^2}\, du \\
&\le 1 - \frac{y^2 \sigma_n^4}{2(1 + 2c)} \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2t_n r/(y\vee 1)}}^{\sqrt{2t_n r/(y\vee 1)}} u^4 e^{-u^2/2}\, du,
\end{aligned}$$

where the last inequality is due to change of variable, (3.5) and (3.6). Since $t_n \to \infty$, by choosing $M \gg 1/r$, for $n \gg 1$ and $|y| \le (r/M)t_n$,

$$|\mathrm{Re}\, I_n(y)| \le 1 - \frac{3y^2 \sigma_n^4}{2(1 + 3c)}.$$

On the other hand, by Lemma 4.1, (3.5) and (3.6), for $n \gg 1$ and $y \in \mathbb{R}$,

$$\begin{aligned}|\operatorname{Im} I_n(y)| &\le z e^{-h(t_n)} \int_{-\sqrt{r/y}}^{\sqrt{r/y}} |\sin(yu^2)| e^{-t_n u^2} f(z + zu)\, du \\ &\le |y| z e^{-h(t_n)} \int u^2 e^{-t_n u^2} f(z+zu)\, du \\ &\le \sqrt{1+c}\, |y| \sigma_n^2 .\end{aligned}$$

As a result, for $n \gg 1$ and $|y| \le (r/M) t_n$,

$$|I_n(y)| \le \sqrt{\left[1 - \frac{3y^2\sigma_n^4}{2(1+3c)}\right]^2 + (1+c) y^2 \sigma_n^4} \le 1 - \frac{y^2\sigma_n^4}{1+c}, \tag{4.13}$$

On the other hand, for $n \gg 1$,

$$\begin{aligned}|J_n(y)| &\le z e^{-h(t_n)} \sup f \int_{|u| \ge \sqrt{r/y}} e^{-t_n u^2}\, du \\ &\le (1+c) A P(|Z| \ge \sqrt{2 t_n r/y}),\end{aligned}$$

where $Z \sim N(0,1)$ and $A = \sup f / f(z) < \infty$. Recall that $P(|Z| \ge x) \sim \sqrt{2/\pi} x^{-1} e^{-x^2/2}$ as $x \to \infty$. Therefore, for $M \gg 1/r$ and $|y| \le (r/M) t_n$, $|J_n(y)| \le e^{-t_n r/y} \le (y/t_n)^2/20 \le y^2 \sigma_n^4/4$.

Combining the bounds for $I_n(y)$ and $J_n(y)$,

$$|\phi_n(y)| \le 1 - \left(\frac{1}{1+c} - \frac{1}{4}\right) y^2 \sigma_n^4 \le e^{-y^2\sigma_n^4/2}, \quad |y| \le (r/M) t_n. \tag{4.14}$$

To verify (4.10) holds for any $\delta > 0$ and $n_0 = n_0(\delta) \gg 1$, by (4.4),

$$\left|E[e^{itY_n}]\right| = \left|\frac{G_n(t_n + it/\sqrt{\Lambda_n''(t_n)})}{G_n(t_n)}\right| = \left|\phi_n\left(\frac{t}{\sqrt{\Lambda_n''(t_n)}}\right)\right|^n .$$

Then, letting $y = t/\sqrt{\Lambda_n''(t_n)}$, by (4.14), for $n \gg 1$ such that $(r/M) t_n \ge \delta$,

$$\left|E[e^{itY_n}] \mathbf{1}\left\{|t| \le \delta \sqrt{\Lambda_n''(t_n)}\right\}\right| = |\phi_n(y)|^n\, \mathbf{1}\left\{|y| \le \delta\right\} \le e^{-ny^2\sigma_n^4/2}.$$

By (4.6), the right hand side is no greater than $e^{-t^2/9}$, which proves (4.10).

To verify (4.11), fix $\delta > 0$ and first let $\lambda \le r/M$. Then by (4.14),

$$\sup_{\delta \le |y| \le \lambda t_n} \left|\frac{G_n(t_n + iy)}{G_n(t_n)}\right| = \sup_{\delta \le |y| \le \lambda t_n} |\phi_n(y)|^n \le e^{-\delta^2 n \sigma_n^4/2}.$$

Since $n\sigma_n^4/\log n \to \infty$, the right hand side is $o(1/\sqrt{n})$. On the other hand, by (4.6), $t_n \sqrt{\Lambda_n''(t_n)} \sim \sqrt{n/2}$. Thus (4.11) holds.

Finally, let $\lambda > \eta := r/M$. From the above proof, it suffices to bound

$$\sup_{\eta t_n \le |y| \le \lambda t_n} \left| \frac{G_n(t_n + iy)}{G_n(t_n)} \right| = \sup_{\eta t_n \le |y| \le \lambda t_n} \left| \frac{\int e^{-(t_n+iy)u^2} f(z+zu)\, du}{\int e^{-t_n u^2} f(z+zu)\, du} \right|^n.$$

By change of variable $u = x/\sqrt{2t_n}$ and letting $\theta = y/t_n$,

$$\frac{\int e^{-(t_n+iy)u^2} f(z+zu)\, du}{\int e^{-t_n u^2} f(z+zu)\, du} = \int e^{-i\theta x^2/2} g_n(x)\, dx,$$

$$\text{where} \quad g_n(x) = \frac{e^{-x^2/2} f(z + zx/\sqrt{2t_n})}{\displaystyle\int e^{-x^2/2} f(z + zx/\sqrt{2t_n})\, dx}.$$

For $y \in \mathbb{R}$ with $|y| \le \lambda t_n$, $\theta \in [-\lambda/2, \lambda/2]$. By the continuity of f at z and $f(z) > 0$, $g_n(x) \to e^{-x^2/2}/\sqrt{2\pi}$ pointwise. So by dominated convergence

$$\int e^{-i\theta x^2/2} g_n(x)\, dx \to \frac{1}{\sqrt{1+i\theta}}$$

uniformly for $\theta \in [-\lambda/2, \lambda/2]$. Given $c > 1$, for all $n \gg 1$,

$$\left| \int e^{-i\theta x^2/2} g_n(x)\, dx \right| \le \frac{c}{|\sqrt{1+i\theta}|} = \frac{c}{(1+\theta^2)^{1/4}}.$$

It follows that

$$\sup_{\eta t_n \le |y| \le \lambda t_n} \left| \frac{G_n(t_n + iy)}{G_n(t_n)} \right| \le c^n \left(1 + \frac{\eta^2}{4}\right)^{-n/4}.$$

By choosing $c \approx 1$, the right hand side is α^n for some $\alpha \in (0,1)$, and hence is $o(1/(t_n \sqrt{\Lambda_n''(t_n)}))$. The entire (4.11) is thus verified. ☐

Appendix. To prove (2.3), let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Let $u_0 = (1/\sqrt{n}) \sum_{i=1}^n e_i$, $u_1, \ldots, u_{n-1} \in \mathbb{R}^n$ be an orthonormal basis. Under $\{u_i\}$, the coordinates of $\sum_{i=1}^n X_i e_i$ are $Y_0, Y_1, \ldots, Y_{n-1}$, with $Y_0 = \sqrt{n}\bar{X}$. Then $\bar{X}$ and $Y = (Y_1, \ldots, Y_{n-1})$ have joint density

$$g(t,y) = \sqrt{n} \prod_{i=1}^n f(x_i), \quad \text{with} \quad \sum_{i=1}^n x_i e_i = \sqrt{n} t u_0 + \sum_{i=1}^{n-1} y_i u_i, \ y \in \mathbb{R}^{n-1}.$$

On the other hand, $V \sim |Y|/\sqrt{n}$ and $\xi := Y/|Y| \in B_{n-1} = \{x \in \mathbb{R}^{n-1} : |x| = 1\}$ almost surely, where $|\cdot|$ stands for the L^2-norm. Let ν be the uniform measure on B_{n-1}. By $Y = \sqrt{n} V \xi$, $g(t,y)$ and the joint density $k(t,s,z)$ of $(\bar{X}, V, \xi)$ with respect to $dt\, ds\, \nu(dz)$ are related via

$$g(t,y) = \frac{k(t,s,z)}{(\sqrt{n})^{n-1} s^{n-2}}, \quad \text{with} \ \ y = \sqrt{n} s z.$$

Since $\phi : z \to \omega = \sum_{i=1}^{n-1} z_i u_i$ is an isometric mapping from B_{n-1} to U_n, $\phi^* \nu$ is the uniform measure on U_n. Eq. (2.3) then follows from

$$\begin{aligned} h(t,s) &= \int k(t,s,z)\,\nu(dz) \\ &= (\sqrt{n})^{n-1} s^{n-2} \int g(t, \sqrt{n} s z)\,\nu(dz) \\ &= (\sqrt{n})^{n} s^{n-2} \int_{B_{n-1}} \prod_{i=1}^{n} f(t + \sqrt{n} s \omega_i)\,\nu(dz) \\ &= (\sqrt{n})^{n} s^{n-2} \int_{U_n} \prod_{i=1}^{n} f(t + \sqrt{n} s \omega_i)\,(\phi^*\nu)(d\omega). \end{aligned}$$

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